Mathematics History in the Geometry Classroom

An Honor Thesis (HONRS 499)

by

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Muncie, Indiana

April, 1993

Graduation: May 8, 1993

Abstract

This thesis is a collection of history lessons to be used in a mathematics classroom, specifically the geometry class, but many of the topics and persons discussed will lend themselves to discussion in other mathematics class. There are fourteen lessons to correlate with chapters of most geometry texts. The introduction contains several useful suggestions for integrating history in the classroom and use of the material. The goal of these lessons is to add interest in enthusiasm in an area that generally receives negative reactions from many students--math.

"In most sciences one generation tears down what another has built, and what one has established another undoes. In mathematics alone each generation builds a new story to the old structure." -Hermann Hankel (Boyer and Merzbach, p. 619).

Why use history? As Posamentier and Stepelman put it, in reference to the mathematics classroom ". . . to breathe life into what otherwise might be a golem." Using the lives, loves, successes, and failures of the people that created mathematics gives interest to a sometimes boring topic. History can be integrated into the classroom in a light and lively way (Posamentier and Stepelman, p. 156). The use of history can attract student interest and enthusiasm as it did with me.

The following are ways to use history in the classroom:

-mention past mathematicians anecdotally
-mention historical introductions to concepts that are new to students
-encourage pupils to understand the historical problems which the concepts they are learning are answers
-give "history of mathematics" lessons
-devise classroom/homework exercises using mathematical texts from the past
-direct dramatic activity which reflects mathematical interaction
-encourage creation of poster displays or other projects with a historical theme.
-set projects about local mathematical activity in the past
-use critical examples from the past to illustrate technology/methods
-explore past misconceptions/errors/alternative views to help understand and resolve difficulties for today's learners
-devise a pedagogical approach to a topic in sympathy with its historical development
-devise the ordering and structuring of topics within the syllabus on historically informed grounds
(Fauvel, p. 5)

Using the chronological table constructed by Adele, in the September, 1989 issue of the <u>Mathematics Teacher</u>, or one you have the students construct, and his suggestion of using pictures, is another approach to bring history into the classroom. Photocopy pictures of the persons/topics you are discussing enlarging them if necessary. Then make these pictures into transparencies that the students can view while the topic/person is being discussed. This will give the students an image to associate with the idea or person and should make it "more realistic" for them (Adele, p. 460).

Another way to add interest is to have the students write papers concerning some mathematical idea or person from the past. This way they can read the anecdotes and lives of mathematical persons from the past. Using a variation on the idea discussed by Klein in the February, 1993 issue of the Mathematics Teacher, use a bulletin board. Have students watch the paper, journals, and other sources for articles discussing mathematics. Maybe there was a new discovery, a new approximation for pi, a new proof for a theorem, or a lost document found. The article by Lamb in the same issue is a fine example to use to aid in the discussion of the history of mathematics. Any of these ideas, plus many more, will aid in the discussion of mathematics beyond the typical expectations of most students and hopefully lead to a positive attitude about mathematics.

This guide contains topics to correspond with the chapters in the geometry text by Jurgensen, Brown, and Jurgensen published by Houghton Mifflin Company. Each topic is somehow related to the chapter and its contents. A quotation is included with each section to add to the information and discussion. The first paragraph of each section reinforces the relationship and explains why the topic is appropriate for discussion within that chapter. The dates used with the persons or topics section are from Adele's article or the Burton book.

The accuracy of dates and information depends on the topic and time period being discussed. Dates and numbers vary from source to source. This is to be expected with historical information from the distant past. Boyer and Merzbach emphasize that much information is based on tradition, conjecture, and inferences because of the loss of documents. There is more known about the Babylonian algebra and Egyptian geometry from 1700 B.C. than the Greek mathematics of 600 to 450 B.C. because of documents missing from the Greek period. But we trust tradition and what was written about the past to "fill in" the missing information (Boyer and Merzbach, p. 68-69).

After viewing the lessons, one will notice that no women appear as the focus of any chapter. With concern for female involvement in mathematics, it should be noted that history is not absent of women but few great mathematicians in history have been women. In fact, some contribute the avoidance of mathematics by women to the first prominent woman mathematician, Hypatia (370? - 415).

Hypatia attracted many students with her outstanding ability, modesty, and beauty. Christians in Alexandria were angered by her pagan learnings and in March of 415, they dragged Hypatia from her chariot to the steps of a Christian church. There they stripped her and brutally murdered her with oyster shells. To be sure they had accomplished their goal, they burned her remains (Dodge, p. 210).

Other women of interest may include Sophie Germain or Maria Agnesi. Sophie Germain (1776-1831) was praised as being one of the promising young mathematicians of the future by Lagrange

(1736 - 1813) (Burton, P. 509). "The Witch of Agnesi," the name given to a curve Pierre de Fermat (1601 - 1665) studied when working with area, refers to Maria Agnesi, (1781 - 1799). She was an Italian mathematician proficient in several languages by the age of thirteen (Lowe, p. 210). There are several other women worth mentioning, but that is another paper!

One should note that the information in this guide is not necessarily arranged in the form of formal prose, it is merely a collection of stories, facts, myths, etc that one should find useful for implementing history and thus interest in the geometry classroom or any mathematics classroom.

Chapter One Points, Lines, Planes and Angles

Origins of Geometry

When did geometry start? Where did geometry start? Why did geometry start? These are questions that will yield different answers from different persons, but will add some background and interest to the geometry class.

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world." -Lobachevsky (Boyer and Merzbach, p. 597).

Most will agree that geometry began before recorded history. Some believe geometry developed in Egypt and Babylonia. There it resulted from the desire of priests to construct temples and of kings that wanted land surveyed for taxes. The first methods were crude and intuitive, but met the desired needs (Posamentier and Stepelman, p. 156). In fact, some term the intuitive beginnings of this area of mathematics as subconscious geometry (Eves, p. 165).

According to those believing in this development human ability to recognize physical form and to compare shapes and sizes led to the beginning of geometry. Everyday things in life lead to the need for geometry and its ideas, but many used it unknowingly. Persons needed to bound land, thus arose distance and geometric figures. Buildings gave rise to ideas such as vertical, parallel, and perpendicular. The sun, the moon, seeds, fruits, and flowers led to the idea of curves, surfaces, and solids. Again observation shows one the idea of symmetry, via the body, seeds, and leaves. The need to hold liquids and other items led to the development of volume. These are just a few of the ways that those believing in subconscious geometry explain the beginning of this field (Eves, p. 166).

Those supporting subconscious geometry, name the next development as "scientific geometry." This occurred when the practical needs for geometry came to exist and rules began to be established (Eves, p. 167).

In the Nile River Valley of Egypt there were the "rope-stretchers" that Herodotus (c. 485 - 430 B.C.), the Greek historian, described. The King Sesostris divided the land and gave equal parts to all Egyptians. On this land the king collected yearly taxes. For those persons having land bordered by a river, the rope-stretchers would determine what part was torn away and the tax would be decreased. Herodotus said that this action appears to him to be the way geometry originated (Burton, pp. 56 - 57). The use of surveying was also found in other river valleys, including the Tigris and Euphrates of Mesopotamia, the Indus and Ganges in Asia, and the Hwang Ho and Yangtze in eastern Asia (Eves, p. 168). This idea of surveying is consistent with the meaning of the word geometry, "measurement of the earth" this reinforces this belief in the origin (Eves, p. 167).

We find a differing view from Aristotle (384 - 322 B.C.). Aristotle believed that geometry originated with the priestly leisure class. These persons worked with geometry for the sheer enjoyment of doing mathematics and for ritualistic desires. In India there were the **Sulvasutras**, or "rules of the cord." These were relationships used to construct altars and temples, which supports Aristotle's idea about using geometry for rituals (Boyer and Merzbach, pp. 6 - 7).

Over time observations were made that led to the development of properties and relationships concerning various objects. Practical geometric problems were ordered into groups that could be solved using the same general idea. These ideas led to laws and rules about geometric ideas. Induction, trial and error, and empirical procedures were used to discover geometric results. These results led to rule-of-thumb and laboratory procedures to be used when working geometric problems (Eves, p. 167).

After this, many problems were created and solved using geometry. Persons added to their knowledge and made conjectures concerning rules and equations. Eventually we see written records of work with geometry. Probably the first big name is Thales (c. 640 - 550 B.C.) in chapter two, followed by Pythagoras (c. 560 - 480 B.C.) in chapter eight and Plato (427 - 384 B.C.) in chapter six. Then there is Euclid (330 - 270 B.C.) discussed in chapter three who organized much of what was known at his time. These are just a few of the outstanding contributors to geometry and its development (Lightner, pp. 15 - 17).

Chapter Two Deductive Reasoning

THALES OF MILETUS
(c. 640 - c. 550 B.C.)

Around 600 B.C. deductive reasoning first appeared. It seems that a merchant, Thales, was the first to attempt a proof of any type (Dodge, p. 1).

> "To Thales . . . the primary question was not **What do we know**, but **How do we know it.**" -Aristotle (Boyer and Merzbach, p. 51).

Thales was born in Miletus, a city of Ionia during a time when a Greek colony flourished on the coast of Asia Minor. He spent his early years in commercial ventures. His travels appear to have been where he learned geometry from the Egyptians, and astronomy from the Babylonians (Burton, p. 93). He was considered to be one of the "seven wise men" of antiquity (Eves, p. 171) and is honored today as the man who always said, "Prove it!" (Posamentier and Stepelman, p. 156).

Thales was considered unusually shrewd in politics and commerce, thus many interesting anecdotes are told about him. Some of these anecdotes follow.

According to Aristotle there was a time when for several years the olive trees did not produce. By using his knowledge of astronomy Thales calculated the next time favorable weather would be present. Knowing the next season would produce a bountiful crop he bought all of the olive presses in the area surrounding Miletus. With control of all of the presses and a plentiful crop, he was able to set his own terms and prices for renting the presses. It is said that since he had proved his point, that it is easy for philosophers to become rich, he was a reasonable man concerning the rental of the presses (Burton, p. 93).

Aesop was known to tell the story of Thales' mules which he used in his mountain salt mine. Thales trained mules to bring salt from the mountain mines to the market. During one trip a mule fell when crossing a stream. Since most of the salt in his load dissolved in the water, the mule had a lighter load for the rest of the trip. Because this was a "clever beast," he would fall in the steam each time he crossed it. To teach this mule a lesson, Thales filled its pack with sponges, this time the mule's load became heavier when he fell in the stream. This is said to have solved the problem of the mule getting his load wet in the stream (Dodge, p. 2).

One evening when Thales was looking at the stars, he tripped and fell into a ditch. An old lady who witnessed this asked him, "How can you see anything in the sky, when you can't even see what is at your feet?" (Dodge, p. 3).

When asked how one could lead a better life, Thales replied that one should "refrain from doing what we blame in others." (Dodge, p. 3).

Thales has been credited with the titles "the first mathematician" and the "father of geometry" because of the credit given to him for contributing the deductive method to the organization of geometry. Before his time the idea of using rigorous proofs to develop theorems was not used. Other ideas credited to Thales include:

1. Every angle inscribed in a semicircle is a right angle.
2. A circle is bisected by its diameter.
3. The base angles of an isosceles triangle are equal.
4. If two straight lines intersect, the opposite angles are equal.
5. The sides of similar triangles are proportional.
6. Two triangles are congruent if they have one side and two adjacent angles respectively equal (Burton, p. 94).

The first proposition stated above is known as the Theorem of Thales, but some question the credit given to Thales for this idea and others, especially the fifth one. The reason for this doubt is that the principles used in a calculation based on the ideas of similar triangles had been known in Egypt and Mesopotamia long before Thales used them (Boyer and Merzbach, p. 54). It is also known that the Babylonians knew that an angle inscribed in a semicircle was a right angle (Boyer and Merzbach,

p. 46). There is no doubt that the Greeks contributed the idea of logical structure to geometry, but was Thales the Greek who did this? (Boyer and Merzbach, p. 55).

Others give Thales credit for the aforementioned relationships in addition to a proof of the Pythagorean Theorem. He is also credited with studying various loci and the fact that the sum of the angles of a triangle equals two right angles. This credits him with introducing the idea of an algebraic identity (Lightner, p. 15). As is the case with much of history, sources are questionable, but Thales will still receive credit for asking why.

Chapter Three Parallel Lines and Planes

EUCLID
(330 - 270 B.C.)

Euclid is known as the most celebrated geometer of all time. He is famous for writing The Elements (Adele, p. 461) and for his parallel postulate which led to the development of non-Euclidean geometries.

Many of Euclid's successors referred to him as "The Elementator!" (Boyer and Merzbach, p. 134).

Little is known about Euclid's life, there is not even a birthplace associated with his name. There is reference to Megara, but the real Euclid of Megara was a student of Socrates, two men hardly interested in mathematics. The Euclid we are discussing is generally referred to as Euclid of Alexandria, where he taught mathematics (Boyer and Merzbach, p. 115 - 116). He was considered a patient and kind teacher who was a modest, fair, and genial man of learning (Lightner, p. 18).

Some anecdotes exist about Euclid, but they yield little information about his personal life. These anecdotes and a poem follow.

King Ptolemy asked Euclid if there was a shorter way of learning geometry rather than via The Elements. It is said that Euclid replied that there is "no royal road to geometry" implying that mathematics is no respecter of persons (Burton, p. 155).

A tale says that a youth began studying geometry with Euclid, when the boy finished the first theorem, he wanted to know what he should get by learning "these things." Euclid insisted that knowledge was worth acquiring for its own sake. But to satisfy the man Euclid called on his servant to give the man a coin, "since he must profit from what he learns" (Burton, p. 155).

Euclid Alone Has Looked on Beauty Bare

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.
-Edna St. Vincent Millay (Survey of American Poetry, p. 285).

Euclid is best known for his development of The Elements, but he also wrote many other books. Topics of these other works include optics, astronomy, music, mechanics, and the conic sections (Burton p. 153). He also gave an interesting proof of the Pythagorean Theorem by proving that the area of a square constructed on the hypotenuse of a right triangle is equal to the sum of the areas of squares constructed on the legs of the triangle (Schacht, McLennan, and Griswold, p. 433).

Euclid used the material developed by previous mathematicians and some of his own to write The Elements. He put all of the discoveries of others into a deductive system based on a set of postulates, definitions, and axioms (Burton, p. 153). This work is made up of thirteen chapters called books. The first six books deal with elementary plane geometry, the next three discuss the theory of numbers, incommensurables is the topic of the tenth book, and solid geometry is the main topic of the last three books (Boyer and Merzbach, p. 120). This book has had more editions, over 2000, in more languages than any other book with the exception of the Bible (Posamentier and Stepelman, p. 156).

Euclid's fifth postulate caused many questions and attempted proofs, and it eventually led to the discovery of the non-Euclidean geometries. His version of the postulate states:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles (Kay, p. 340).

Because many persons believed that Euclidean geometry was the only possible geometry there were many attempts to prove that the fifth postulate was not a logical consequence of the other four. Those making attempts include Ptolemy (c. 85 - 165) and Proclus (c. 410 - 485) (Mitchell, p. 77) and many others who arrived at proofs, but discovered they had assumed something equivalent to the fifth postulate in their work. In 1733, Saccheri attempted to prove the postulate indirectly. Since he could not believe Euclid's geometry was wrong, he made an invalid conclusion that contradicted an assumption, in his mind proving the postulate (Kay, p. 12).

It is said that Gauss (1777 - 1855) was the first to realize that non-Euclidean geometries existed, but he did not publish materials because he did not want to cause a stir. Thus, J. Bolyai (1802 - 1860) a Hungarian, and a Russian, Lobachevsky (1793 - 1856), were the first to publish material on a non-Euclidean geometry (Ballard, p. 174). Their geometry is known today as hyperbolic geometry (Kay, p. 13). Riemann (1826 - 1866) assumed that any two lines will intersect and thus found elliptical geometry (Schacht, McLennan, Griswold, p. 91). Because of Euclid's fifth postulate, the non-Euclidean geometries came to exist. These geometries added many interesting and new ideas to the field of mathematics.

Chapter Four Congruent Triangles

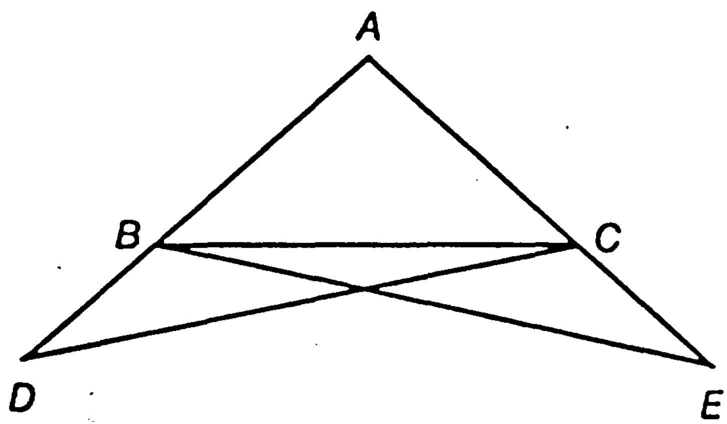
PONS ASINORUM
Middle Ages

Proposition 5 of Book I in <u>The Elements</u> of Euclid states:

"In isosceles triangles the angles at the base are equal to one another and if the equal straight lines be produced further, the angles under the base will be equal" (Somers, p. 219).

In the proof of this proposition, Euclid proved two triangles congruent, thus the desired angles equal, therefore the base angles of an isosceles triangle equal.

The diagram and proof used by Euclid follow.



Given $\triangle$ ABC with $\overline{AB} = \overline{AC}$.
Extend $\overline{AB}$ and $\overline{AC}$ through B and C, respectively, to points D and E, so that $\overline{BD} = \overline{CE}$.
Therefore, $\triangle$ ADC = $\triangle$ ABE so that $\angle$ D = $\angle$ E and $\overline{DC} = \overline{BE}$.
Then $\triangle$ BDC = $\triangle$ BCE, so that $\angle$ DBC = $\angle$ ECB.
Therefore, $\angle$ ABC = $\angle$ ACB
Q.E.D. (Posamentier and Stepelman, p. 156).

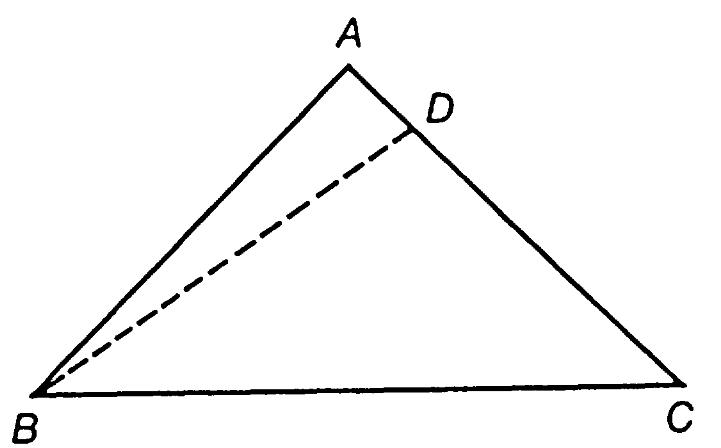
This proposition of Euclid's was called **elefuga**, a medieval term meaning "the flight of the fools." This was usually the point when students would give up learning geometry, when they were not capable of understanding this idea and proof. Another interpretation says that the diagram which Euclid used in his proof looked like a trestle-bridge so steep that a horse could not climb it, but a sure-footed animal, such as an ass, could climb it. Thus only the sure-footed student could go beyond this part of geometry (Burton, p. 160 - 161).

The "bridge of asses (fools)" was used in the Middle Ages as a test for students studying geometry to further their education in geometry. It separated the weak from the better (Posamentier and Stepelman, p. 157).

Isosceles is derived from the Greek words "isos" and "skelos." Isos means equal and skelos means legs, thus we have equal legs as the meaning for isosceles, which is exactly what an isosceles triangle has (Posamentier and Stepelman, p. 156).

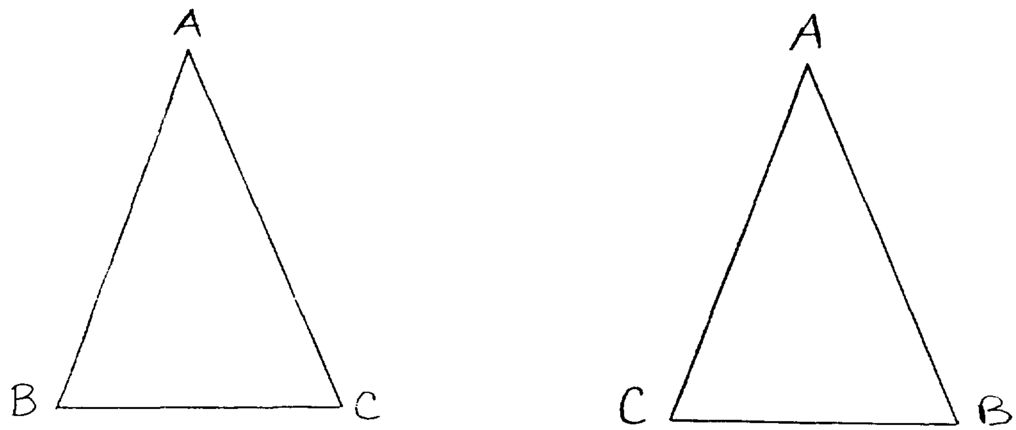
Thales is given credit for discovering this proposition. It is thought that he attempted a formal proof for it, but we do not know for sure. Others, including Proclus (c. 410 - 485), also tried to prove this. His proof, and others, are related to one of Pappus (c. 300). The proof of Pappus involved the picking up and turning over the triangle in order to place the triangle on itself. But questions arose as to the ability to pick up the triangle and leave it there at the same time (Somers, p. 219).

Euclid also proved the converse of this theorem, that given a triangle with two angles equal, the sides opposite them are also equal. Following are the diagram and an indirect proof (reductio ad absurdum) which he used.



Given $\triangle ABC$, where $\angle B = \angle C$.
Assume $\overline{AB} = \overline{AC}$, and let $\overline{AC} > \overline{AB}$.
Mark off point D on AC such that $\overline{DC} = \overline{AB}$.
Then $\triangle DCB = \triangle ABC$. This is impossible since it would make $\angle DBC = \angle ABC$, so $\overline{AB} = \overline{AC}$. (Posamentier and Stepelman, p. 157)

Below is the proof used by Pappus of Alexandria. His proof is considered easier and used no auxiliary lines. He uses the fact that the side-angle-side propostion does not state that the triangles must be distinct.



Given the isosceles triangle ABC, where $\overline{AB} = \overline{AC}$ think of it in two ways: as $\triangle$ ABC and $\triangle$ ACB.

Thus, there is a correspondence between $\triangle$ ABC and $\triangle$ ACB with vertices A, B, and C corresponding to vertices A, C, and B, also $\overline{AB} = \overline{AC}$, $\overline{AC} = \overline{AB}$, and $\angle$ BAC = $\angle$ CAB.

Thus the two triangles are congruent by SAS

Since the triangles are congruent, all the parts in one triangle are equal to the corresponding parts in the other triangle Thus, $\angle$ ABC = $\angle$ ACB, as desired (Burton, p. 162).

Most text books prove this theorem by constructing an angle bisector through the vertex angle. But many "purists" frown upon this method because they feel it introduces the angle bisector prematurely (Posamentier and Stepelman, p. 156). Thus the reason for using the other previously mentioned proofs or other historically used proofs is to add variety to the class.

Chapter Five Quadrilaterals

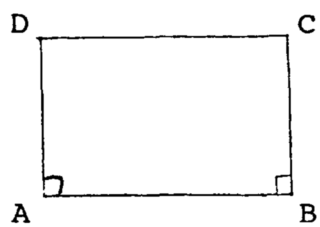
SACCHERI and LAMBERT

A look at these two persons will give the students a different viewpoint concerning quadrilaterals having right angles and the use of these quadrilaterals in the development of non-Euclidean geometries.

"It is the glory of geometry that from so few principles, fetched from without, it is able to accomplish so much."
-Isaac Newton (Burton, p. 151)

Girolamo Saccheri (1697 - 1733) appears to be the first to have studied the logical consequences of an actual denial of the fifth postulate of Euclid. He was a Jesuit priest who taught in various colleges in Italy. He was considered a brilliant teacher with a remarkable memory. Saccheri wrote Logica Demonstrativa, a work on logic. The deductive power of the indirect proof was very interesting to this man as was Euclid's fifth postulate. He spent many years of his life working on this postulate. In 1733, he published a treatise titled Euclid Vindicated of Every Blemish. This work focused on the use of what is now called the Saccheri quadrilateral (Burton, p. 529).

The Saccheri quadrilateral has sides AD and BC equal and both perpendicular to the base AB.

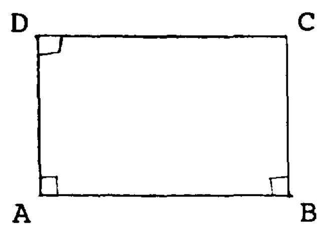


By using congruent triangles, Saccheri was able to prove that the summit angles, angles C and D, are equal. Thus he deduced there were three possibilities for these angles (1) they are greater than ninety degrees, the hypothesis of the obtuse angle, (2) they are less than ninety degrees, the hypothesis of the acute angle, or (3) they equal ninety degrees, the hypothesis of the right angle. By showing hypotheses one and two lead to absurdities, he felt that indirect reasoning would establish hypothesis three (Boyer and Merzbach, p. 487).

He assumed a straight line is infinitely long to eliminate the hypothesis of the obtuse angle (Boyer and Merzbach, p. 487). Next Saccheri attempted to find a contradiction with the acute angle hypotheses (Burton, p. 530). He proved theorem after theorem with this assumption. Since he was so determined to find a contradiction, he forced one to occur. He treated a point at infinity as if it were a point of the plane. He concluded that two distinct lines that meet at an infinitely distant point can both be perpendicular at that point to the same line. This he viewed as a contradiction to Euclid's Proposition 12 which says that there is a unique perpendicular to a line at each point of that line. He said that "The Hypothesis of the Acute Angle is absolutely false, being repugnant to the nature of a straight line." (Burton, p. 531). Little did Saccheri know that he was developing a logical non-Euclidean geometry!

Johann Heinrich Lambert (1728 - 1777) was a Swiss-German writer who wrote on various topics, some mathematical and some not. When asked by Frederick the Great in which science he was most proficient, Lambert said, "All." It is believed that if this man of great ability had focused on fewer fields of science, that he would be better known (Boyer and Merzbach, p. 514). He is probably most remembered for his proof of the irrationality of pi made in 1761 (von Baravalle, p. 152).

In his work, <u>Theorie der Parallellinien</u> published by friends in 1788 after his death, he used the indirect proof method as Saccheri did, but he began with a Lambert quadrilateral. This is a quadrilateral with three right angles, the fourth is either right, acute, or obtuse.



As with the Saccheri quadrilateral, the sum of the angles of a triangle came into play. If the angle was obtuse, the angle sum for a triangle is greater than two angles; acute, the sum was less than two right angles (Boyer and Merzbach, p. 514). But he also considered the extent to which the sum differs from two right angles and its proportion to the area of the triangle.

The obtuse angle case was similar to a theorem in spherical geometry, thus he speculated that there may be a theorem in which the acute angle case would correspond (Boyer and Merzbach, p. 514). Lambert realized that he had not reached a contradiction with the acute angle hypothesis. In this new geometry, he noticed that the sum of the angles of a triangle increase when the area decreases, something no one had discovered before him (Burton, p. 533). He was so close to discovering a non-Euclidean geometry!

Chapter Six Inequalities in Geometry

PLATO
(427 B.C. - 348 B.C.)

This man had an immense influence on the course of mathematics by founding and leading his famous academy in Athens (Adele, pp. 461 - 462). Above the door of his classroom in his academy in Athens was written:

"Let no man ignorant of geometry enter here."

This was a tribute to the Greek convictions that through the spirit of inquiry and strict logic one could understand man's place in an orderly universe (Burton, p. 90). Plato was a disciple of Socrates (469 - 399 B.C.). Who left Athens in haste after Socrates was sentenced to drink poison. He traveled through Egypt, Sicily, and Italy after leaving Athens. In Italy, he became aware of the ideas and ways of the Pythagoreans. Many contribute his appreciation of the universal value of mathematics to this exposure. Upon his return to Greece, he was sold as a slave by the captain of the ship upon which he was traveling. Luckily friends ransomed him (Burton, p. 145).

Around 387 B.C. he returned to Athens as a philosopher. In a grove outside of Athens, he founded his school. It was built on land that had belonged to the hero Academos, thus the name grove of Academia, and the name Academy for the school. As was the tradition of the time the school was a religious brotherhood. It was dedicated to the worship of the Muses. It even had chapels dedicated to these divinities. For 900 years this school was the intellectual center of Greece. The Christian Emperor

Justinian closed it in 529 A.D. because of religious differences. (Burton, p. 145).

Socrates had no interest in mathematics, in fact he probably influenced it negatively. So how did Plato become involved with it? Many give this honor to Archytas (428 - 347 B.C.) a friend of Plato. It is even suggested that during a visit to this friend in Sicily in 388 B.C., Plato learned of the five regular solids. He wrote of these solids in a dialogue with Timaeus of Locri, a Pythagorean, entitled Timaeus. Because of the way that he applied the regular polyhedra to the explanation of scientific phenomena in this dialogue, they have been called "cosmic bodies" or "Platonic solids" (Boyer and Merzbach, p. 97).

Plato viewed the faces of the solids as more than simple triangles, squares, and pentagons. He used the triangle to describe the faces. He considered the faces of the tetrahedron to be made up of six smaller right triangles formed by altitudes of the equilateral triangular faces. Thus the regular tetrahedron was made up of twenty-four scalene right triangles in which the hypotenuse is double one side. The regular octahedron is made up of forty-eight of these triangles and the isocahedron, one hundred and twenty of them. The cube, or hexahedron, is composed of twenty-four isosceles right triangles formed by the diagonals of the squares. The dodecahedron was made up of three hundred and sixty scalene right triangles, each face containing thirty right triangles formed by the diagonals and medians of the pentagons. He felt that this solid was special and he considered it representative of the universe (Boyer and Merzbach, pp. 99 - 100). Because of his limited contributions, perhaps no original ones, to mathematics, Plato has been referred to as the "maker of mathematics." He encouraged mathematicians better than himself to work with the field and make something of it (Bell, p. 26). More specifically, with the restrictions he placed on geometry, it is considered that he made geometry what it is (Bell, p. 32). These restrictions refer to the limit for the tools used in constructions, the unmarked straightedge and the collapsible compass. These tools are the only ones that Plato would allow to be used when constructing figures. These limits led to the many attempts to solve the three great problems of antiquity, squaring the circle, trisecting any angle, and doubling the cube (Retz and Keihn, pp. 192 - 193).

If anything, Plato deserves recognition for the history he gave us about others and ideas through his dialogues, such as Phaedo, about Socrates; Timaeus, concerning the regular solids (Boyer and Merzbach, pp. 96 - 97); and Hippias Major and Hippias Minor about Hippias's (460 B.C. - ?) life and character. Hippias had a vase knowledge of many subjects due to his excellent memory. Once he recited a string of fifty names, in the correct order. His memory and many eccentricities led tot he description of him as an "arrogant boastful buffoon" (Burton, p. 141). Also there should be the recognition for his school in which he made mathematics an essential part of the curriculum (Boyer and Merzbach, p. 100).

Chapter Seven Similar Polygons

KARL FRIEDRICH GAUSS
(1777 - 1855)

Since he is such an outstanding contributor to mathematics and the one who first constructed the seventeen sided regular polygon using Euclidean tools, Gauss deserves mention in any mathematics classroom.

"If others would but reflect on mathematical truths as deeply and as continuously as I have, they would make my discoveries." -Gauss (Bell, p. 254)

Johann Carl Freidrich Gauss was born into a poor family in Brunswick, Germany on April 30, 1777. His father was a hard laborer who was strict and planned for Gauss to follow in his footsteps. But fortunately Friederich Benz, Gauss' maternal uncle, was clever and developed the young Gauss' mind. He used quizzical observations and a mocking philosophy of life to develop the photographic mind of Gauss (Bell, pp. 218 - 219).

When he was three years old, Gauss corrected an error his father had made in a payroll calculation (Burton, p. 510). At twelve years old, he was questioning the foundations of elementary geometry. When he was sixteen he began to look at non-Euclidean geometries (Bell, p. 223).

The story is told that one day his teacher told the class to add all the numbers from one to one hundred, to keep them busy. When finished, they were to lay their slates on a table. Almost immediately, Gauss laid his slate on the table and said, "There it is." Upset with him, the teacher looked scornfully at Gauss, but later discovered that he was the only student with the correct answer and that was the only thing written on the slate (Boyer and Merzbach, p. 558).

Gauss admmitted he recognized a pattern

$1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, . . . $50 + 51 = 101$.

Since there are 50 of these pairs that add to 101, the sum of all the numbers must be 50 X 100 = 5050 (Burton, p. 510).

After this action, his teacher admitted that he could teach Gauss no more (Dodge, p. 265) and mentioned this genius to the Duke of Brunswick, Ferdinand, who became Gauss' patron for many years. When he entered the University of Gottingen, Gauss was torn between mathematics and classical languages. What made him choose mathematics? On March 30, 1796, he constructed a regular polygon with seventeen sides using only the compass and straightedge (Burton, p. 511).

This was just one of the many discoveries Gauss made in the area of mathematics. Unfortunately, he insisted on extremely concise publications and generally did not publish many of his discoveries because he did not have adequate time to develop them and their proofs to his desired level (Burton, p. 549). Some feel this attitude may have developed from the refusal by the French Academy of Sciences to publish his first work, <u>Disquistiones Arithmeticae</u>. But from his paperwork and diary, published after his death, it was learned that Gauss had known many important mathematical ideas. Had he only published them, his reputation and the advancement of mathematics would have been phenomenal (Bell, pp. 228 - 229).

He was hesitant to publish his work concerning the development of non-Euclidean geometry, because of its lack of perfection and because he did not want to cause controversy or an attack on himself. But he did write to friends about his work concerning the parallel postulate. He considered it independent of the other Euclidean axioms and decided he could use a contradictory axiom to build a new geometry equally as logical as Euclid's. Assuming the sum of the angles of a triangle is less that 180 degrees, he developed a different geometry. He wrote in a letter that he developed this geometry to his own satisfaction and may make it public in the future. But in another letter decides that he will not publish his views for fear of attack from others (Burton, pp. 550 - 551).

The interests and work of this "Prince of Mathematics" covered many areas. Included in his focus was astronomy; geodesy; the theories of surfaces and conformal mapping; mathematical physics, specifically electromagnetism, terrestrial magnetism, and the theory of attraction according to Newton; analysis situs; and the geometry associated with functions of a complex variable (Bell, p. 263).

Upon the death of his patron, the Duke of Brunswick, during the battle of Jena in 1806, Gauss became concerned about finances. His friends, eager to keep Gauss in Germany, obtained him the position of director at a newly built observatory at the University of Gottingen. This is where he lived until his death. (Burton, p. 513).

Gauss escaped death one day in June of 1854. While he was traveling to see a railroad being built. The horses bolted and he was thrown from his carriage. He was unhurt, yet badly shocked. But he was able to witness the opening ceremonies of this railroad. Unfortunately, the new year of 1855 brought pain and suffering to Gauss. He had an enlarged heart, shortness of breath, and symptoms of dropsy. Despite his illness Gauss worked when he was able. A cramped hand ruined his handwriting that had been considered beautiful. His last letter was to Sir David Brewster, concerning the discovery of the electric telegraph. Early on February 23, 1855, Gauss died (Bell, pp. 268 - 269).

Gauss requested that a regular seventeen sided polygon be carved on his tombstone. But the stonemason carved a seventeen point star because he was afraid the polygon would look like a circle (Burton, p. 511). A monument erected at his place of birth in Brunswick, Germany, is a regular seventeen sided polygon. This was to recognize his greatest achievement (Posamentier and Stepleman, p. 159).

Chapter Eight Right Triangles

PYTHAGORAS
(c. 560 - 480 B. C.)

Pythagoras' name is attached to the Pythagorean Theorem even though there is evidence that this theorem was known before his time. But it is said to be the Pythagoreans who developed the first proof of this theorem, thus the attached name (Hirschy, p. 215). The investigation of the Pythagorean Theorem and proofs of it is another endeavor that, like the Pythagoreans, some students may find interesting.

"Number rules the universe." -The Pythagoreans (Bell, p. XV)

Pythagoras was born at Samos, one of the Dodecanese islands. He traveled to Egypt, Babylon, and probably India. During these trips he gained information pertaining to mathematics, astronomy, and religion. After returning to Magna Graecia, (on the southeast coast of today's Italy) he established the secret Pythagorean School (Boyer and Merzbach, p. 55 - 56).

What was this school like? According to what was know, this society met in secret and members were very superstitious. It was said to be a "brotherhood knit together with secret and cabalistic rites and observances committed to the study of philosophy, mathematics, and natural science" (Eves, p. 172). Other authors have commented on their beliefs.

It is said that they refused to eat beans, drink wine, pick up anything that had fallen, or stir a fire with an iron. These persons believed that one's soul could leave the body either temporarily or permanently. They also felt they could inhabit the body of another person or animal. Because of these feelings, they would not eat fish or meat for fear it may contain a friend's soul. The would only kill an animal as a sacrifice to the gods. They even refused to wear clothing made of wool because it is an animal product. A story says that Pythagoras told a person to quit beating a dog because his friend's soul lived in the dog. How did he know? He recognized him by his voice (Burton, p. 99).

The Pythagoreans believed that all things could be formed by the whole numbers and their ratios. Their official symbol was the pentagram, whose various line segments repeat the "golden ratio" of proportion. Today three to five is commonly used as the golden ratio in art and architecture to make the object pleasing to the observer.

Any discoveries that were made by this society were credited to its founder, Pythagoras. But most of these were kept secret and revealed only to members of the society. When it was shown that the square root of two was irrational, the foundations of this society were destroyed as was the Pythagorean, Hippasus, who revealed this secret (Dodge, pp. 41 - 42). The Pythagorean Hippasus was reported by his brother Pythagoreans to have drown for revealing the irrationality of the square root of two (Dodge, p. 101).

Another important contribution of this society was the production of chains of theorems, leading to the eventual realization that all mathematics needed to be developed from a simple basis set of assumptions (Dodge, p. 42). They developed a theory of proportion, but it was limited to commensurable magnitudes. Using this theory, they deduced properties of similar figures. They were aware of the existence of at least three regular polyhedral solids (Eves, p. 172).

However, the Pythagoreans are most noted for proving the Pythagorean Theorem. It is suspected that persons knew long before this time of this relationship in a right triangle. The Egyptians (2000 B.C.) noted that four squared plus three squared yields five squared on a papyrus fragment. A Chinese text dating to 202 B.C. tells the reader to "break the line and make the breadth three, the length four; then the distance between the corners is five." This text also contained diagrams now associated with proof of the theorem (Hirschy, pp. 215 - 216).

There are conflicting stories concerning Pythagoras' death, was it old age, was it illness, or was it murder? You decide.

Toward the end of his life Pythagoras retired to Metapontum and died there about 500 B.C. He left no written works but his ideas were carried on by his many eager disciples (Boyer and Merzbach, p. 81).

While traveling Pythagoras fell ill. A kind-hearted innkeeper nursed him, but he did not survive. Before he died, he drew the pentagram star on a board and begged the innkeeper to hand it outside. The innkeeper did as Pythagoras wished. Some time later a fellow Pythagorean passed by and noticed the pentagram. The innkeeper told of Pythagoras and the Pythagorean rewarded him for his deed (Burton, p. 99).

Or maybe Pythagoras died when angry persons who did not like his ideas and teachings attacked his school. It is claimed that this mob set fire to this school and he perished in the fire at the age of seventy-two. But his scattered society continued for two more centuries (Dodge, p. 43).

Chapter Nine Circles

PI

The ratio of the circumference of a circle to its diameter and the ratio of the area of a circle to the area of the square on its radius is called pi. Pi is also a ratio for some surface areas and volumes in solid geometry (von Baravalle, p. 148).

I shall risk nothing on an attempt to show the transcendence of pi. If others undertake it, no one will be happier than I at their success, but believe me, my dear friend, this cannot fail to cost them some effort. -Carl Wilhelm Borchardt (Burton, p. 606).

The symbol for pi, π , was introduced in 1739 by Leonhard Euler (Burton, p. 221). He had first used this symbol in a 1731 letter to Goldbach. It had previously appeared in Clavis Mathematicae by Oughtred in 1647. Oughtred used pi to represent circumference (Burton, p. 503). One also finds it in William Jones' A New Introduction to the Mathematics, of 1706 (Boyer and Merzbach, p. 494).

In the history of mathematics, many different and extremely accurate values for pi have been used. It is amazing to note the accuracy used for pi, despite the lack of proofs/development. Following is a discussion of some values which were used and a brief explanation of how these values were obtained. As with most historical items, the details vary, thus different sources list different values.

From the problems in the Rhind Papyrus (1650 B.C.) we were able to determine that the Egyptians used three and one-seventh in their work. Through the examination of Babylonian problems is also how their value of three for pi was determined. But a table discovered in 1936 indicated a second value, three and one-eighth. The Hebrews used the value given in the Old

Testament, three. One can refer to I Kings 7:23 and the bath in the temple of Solomon for this discussion (Burton, p. 58).

Archimedes (287 - 212 B.C.) used polygons inscribed in and circumscribed about a circle to approximate its area. Beginning with a hexagon, because it was easiest to inscribe, he then used regular polygons having 12, 24, 48, and 96 sides. Using the perimeters of the inscribed and circumscribed polygons, he was able to determine that pi was greater than three and ten seventy-firsts but less than three and one-seventh. For a more detailed explanation of the process see Burton (Burton, p. 221). Others using this method include Ptolemy (c. 85 - 165). He used a polygon of 720 sides and a circle with a radius of 60. His approximation for pi is 377/120 or about 3.1416. Francois Viete (1579) used polygons having 393,216 sides to find a value of pi to nine decimal places. Ludolph van Ceulen, in 1610, used a polygon with 2^62 sides to find a value of pi to thirty-five places (von Baravalle, p. 151). Aryabhata (c. 476 - 550) found his value by performing a series of steps to find an approximation for the circumference of a circle with a diameter of 20,000. One was told to add 4 to 100, multiply this by 3, add to the result 62,000. Dividing this result by 20,000 yields his value of 3.1416 (Burton, p. 145) which is very close to the square root of ten, known as the Hindu value (Boyer and Merzbach, p. 241).

Brahmagupta (c. 625), a Hindu, used two different values for pi depending on his need. He considered three as a practical value and the square root of ten as an exact value. His exact value was used during the middle ages by most persons (von Baravalle, p. 151).

Series have been used to find an approximation for pi. A few follow. In 1671, James Gregory a Scottish mathematician and in 1673, Gottfried Leibniz used a series now known as the Leibniz series for their approximations. This series is pi/4 = 1 -(1/3) + (1/5) -(1/7) + (1/9) - (1/11) + ... But some feel this series converges too slowly. Another possible series is Newton's, pi divided by two times the square root of two. His series equals 1 + (1/3) -(1/5) - (1/7) + (1/9) + (1/11) - (1/13) -(1/15). . . (Burton, p. 388).

In 1853, William Shanks completed fifteen years of calculation to give a 707 place approximation for pi. At a later date, an error was found in the 528th place, but his attempt was still an outstanding feat (Burton, p. 590). Computers have been used to calculate pi to many decimal places. In 1949, ENIAC took seventy hours to calculate pi to 2,037 places. In 1958, only one hour and forty minutes were required to obtain 10,000 places. The first 707 places were calculated in just forty seconds. One hundred thousand two hundred sixty-five places were calculated in 1961 with a time of eight hours and forty-three minutes needed. A French computer, CDC 6600, gave a 500,000 place approximation in 1967 (von Baravalle, p. 152 - 153).

With these amazing approximations, how does one know when to stop? Several persons have proved that there is no stopping point. In 1770 and 1794, Lambert and Legendre proved that pi and pi squared are both irrational numbers. This proof did not convince everyone that the age old problem of squaring a circle, constructing a square with an area equal to the area of a given circle, could not be solved, Lindemann was the person to do this. His paper, in 1882, extended the work of Lambert and Legendre and he proved that pi is a transcendental number (Boyer and Merzbach, p. 639). A transcendental number is one that is not algebraic. An algebraic number is a real or complex number that satisfies a polynomial equation with integer coefficients (Fey, p. 83). This finally proved that the quadrature of the circle is impossible with Euclidean tools (Boyer and Merzbach, p. 639) and that the decimal representation for pi extends indefinitely.

"How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics," is an interesting mnemonic one could use to remember an approximation to pi. By counting the number of letters in each word, one can find the digits for pi (Boyer and Merzbach, p. 272 - 273). One might try to find a more appropriate replacement for "alcoholic" before presenting this to students but it is a beginning for this amazing number.

Chapter Ten Constructions and Loci

CONSTRUCTION PROBLEMS AND RESTRICTIONS

Constructions can be viewed from many different viewpoints. There are the limits imposed on constructions, by Plato and others, and there are the "problems of antiquity." Both of these ideas will add interest to daily constructions.

"A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts." -David Hilbert (Burton, p. 289)

To begin with, one should make the distinction between drawing and constructing. Drawing includes the use of a ruler, a protractor, a parallel rule, a draftsman's triangle and T-square, and many other tools. When constructing, one is limited to the unmarked straightedge and compass (Schacht, McLennan, and Griswold, p. 38). Why are these tools the only ones that can be used?

The story is told that Plato (427 - 348 B.C.) made this decision. He was in charge of a Greek school that emphasized the study of plane geometry. It is said that he thought his students would learn to reason better if they were limited to the straightedge and compass when doing constructions (Schacht, McLennan, and Griswold, p. 37).

Not only were the tools limited but there were restrictions placed on their usage. The straightedge was to be used to draw a line through two given points. The compass was to be used to draw a circle with a given center and radius. Neither instrument was to be used to transfer distances, thus the straightedge was not marked in any way and the compass would collapse as soon as either point was lifted off the paper (Burton, p. 130 - 131). The simple construction of an equilateral triangle, with a given segment as one of its sides, by Euclid in his first three propositions in Book I, shows that today's compass and straightedge is equivalent to the collapsible compass and the straightedge of the past (Retz and Keihn, p. 193). In other words, the constructions that are made with today's compass can be made with the collapsible one and vice versa.

The Greeks focused on three particular problems which have come to be known as the famous problems of antiquity (Anderson, p. 197). These problems include the quadrature of the circle or the squaring of the circle, the duplication of the cube, and the trisection of a general angle (Burton, p. 130). (One should point out that these problems can be solved using other tools and curves, but not with Plato's restrictions.)

The quadrature of the circle fails because of the need to construct a line segment whose length is the square root of pi times the radius of the circle. When Lindemann, in 1882, proved the transcendence of pi, this problem was proved to not be constructible using Plato's restrictions (Burton, p. 131).

In order to duplicate the cube, one would need to find the edge of a cube having a volume twice that of a given cube. Where did this problem arise from?

Some claim that when the Pythagoreans (c. 560 - 480 B.C.) were able to double a square or construct a square with twice the area of a given square it seemed only natural to do this type of construction in three dimensions.

Or maybe this "Delian problem" comes from the Athenians appeal to the oracle at Delos in 430 B.C. They desired to know what they could do to fight the devastating plague which was inflicting great suffering on their city and had killed their leader, Pericles. The oracle told them to double the size of the altar of Apollo, which was a cube. Workmen simply constructed a cube whose edge was twice as long as the previous one, because of this error, the story says that the plague became worse. This time citizens went to Plato for help. He told them that "the god has given this oracle, not because he wanted an altar of double the size, but because he wished in setting this task before them to reproach the Greeks for their neglect of mathematics and their contempt of geometry." No one knows what happened after this.

Then again maybe it was King Minos who wanted to erect a tomb to his son Glaucus. He felt that the constructed tomb was too small for his royal son and exclaimed, "You have enclosed too small a space; quickly double it, without spoiling the beautiful (cubical) form." (Burton, p. 134).

Hippocrates of Chios (460 - 380 B.C.) was never able to find the mean proportionals needed to duplicate the cube using Plato's restrictions (Burton, p. 135). In 1637 Descartes finally proved the impossibility of this construction, with restrictions, because of the need of a parabola to solve a cubic and determine the mean proportionals (Anderson, p. 198).

It is suspected that the desire to construct a regular polygon with nine sides is what brought about the angle trisection problem. In order to construct this figure, the trisection of a sixty degree angle was necessary (Habegger, p. 199). It was Pierre Wantzel (1814 - 1848) who was the first to give a rigorous proof of the impossibility of solving the third problem of antiquity. In 1837, he showed the impossibility of trisecting any given angle using only the straightedge and compass (Burton, p. 135).

Plato placed the compass and straightedge restrictions on constructions, yet throughout history others have placed even greater restrictions on the constructions. In 1672 a Danish mathematician, Georg Mohr (1640 - 1697) published <u>Euclides</u>

Danicus. In this work he showed that if one considers a line as known whenever two distinct points on it are known, then one can simply use the compass alone for constructions (Boyer and Merzbach, p. 413). The claim is that he was answering the question, "What is possible using only the compass?" He showed that all of Euclid's constructions were possible (Retz and Keihn, p. 192). His work was ignored and lost until 1928 when a mathematician found his book in an old book store. In 1797, Mascheroni rediscovered the idea that the straightedge was not needed in constructions (Boyer and Merzbach, p. 413). Thus we have the Mohr-Mascheroni method of construction.

Abu'l-Wefa, a tenth century Arab carried the construction restrictions even further. His approach is referred to as the "rusty compass" one. The straightedge is allowed, but the compass being used is fixed at a set opening. Using this method one can again "construct all of Euclid" (Retz and Keihn, pp. 194 - 195).

In 1822, Jean-Victor Poncelet (1788 - 1867) suggested that all Euclidean constructions could be completed using the straightedge alone after a fixed circle was drawn with a compass. Jakob Steiner (1796 - 1863) proved in 1833, that all Euclidean constructions could be completed using this method (Boyer and Merzbach, p. 602). In 1904, Francesco Severi proved that only a small arc of a circle and its center was necessary for straightedge alone constructions (Retz and Keihn, p. 196).

Which method is the best? Each person has his/her own opinion on this topic. Emile Lemoine gave a set of criteria in

1907. The simplicity of the construction is the sum of the numbers of the various simple operations used in the construction (Retz and Keihn, p. 196).

A source for many constructions is Contemporary Geometry by Schacht, McLennan, and Griswold. They include the basic constructions as well as ones involving polygons, tangents, circles and other figures.

Chapter Eleven Areas of Plane Figures
11-8 Geometric Probability

BLAISE PASCAL
(1623 - 1662)

Pascal worked with Fermat, developing probability theory via letters. Had he devoted more time to mathematics instead of turning to religion, who knows what other accomplishments he might have made.

"The theory of probabilities is at bottom nothing but common sense reduced to calculus; it enables us to appreciate with exactness that which accurate minds feel with a sort of instinct for which oft times they are unable to account." -Laplace (Burton, p. 411)

Blaise Pascal was born on June 19, 1623, in France to Etienne and Antoinette (Begone) Pascal. He had two sisters with talents of their own (Bell, p. 73). His father limited his education in order to avoid injuring his health. In fact, mathematics was taboo because Etienne thought Pascal might overstrain himself by using his head. At age twelve, he demanded to know what geometry was. After his father explained it, Pascal was hooked (Bell, p. 74). He began attending informal meetings at the "Mersenne Academy" in Paris with his father (Boyer and Merzbach, p. 402).

Blaise is considered one of the greatest might-have-beens in mathematical history. He had great possibilities as a mathematician, but religion was where he eventually focused his energies (Bell, p. 73). In 1646, the Pascal family began to follow the creed of Jansenism, hating Jesuits. From there Pascal's focus became religion, with a few episodes of mathematics (Bell, p. 79).

Yet, he did make some worthwhile contributions to mathematics before his religious involvement. In 1642, he invented a calculating machine. This machine would do addition and subtraction. It was a large rectangular box with six wheels, designed to do calculations with English currency (Davis, p. 112). He also invented an instrument to trisect an angle (Posamentier and Stepelman, p. 158). In 1654, Pascal began work on Complete Work on Conics, which many believe was an extension of the essay written on the conics at age sixteen. Unfortunately this larger work was never published, but Leibniz (1646 - 1716) saw a copy and made notes concerning it (Boyer and Merzbach, p. 403).

A story is told that one night in 1658, after Pascal had given up mathematics for religion, he could not sleep because of a toothache or illness. To divert his attention from the pain, he studied the cycloid. The pain eased. Pascal saw this as a sign from God that he could study mathematics without upsetting Him. He found areas, volumes, and centers of gravity associated with the cycloid. These findings caused him to create a contest with prizes for those answering his questions concerning the cycloid. Unfortunately only two persons sent answers, thus no prizes were awarded. But his actions did bring interest to the cycloid (Boyer and Merzbach, p. 406).

In 1654, Chevalier de Mere, a member of the French nobility, wrote Pascal concerning some questions about gambling. Interested in these questions, Pascal wrote Pierre de Fermat (1601 - 1665). Letters between these two men developed the basic ideas of the theory of probability (Burton, p. 426). Their letters show the development of the fundamental principles of probability. While trying to take short cuts, Pascal encountered errors which Fermat pointed out and Pascal acknowledged. But overall the men brought "the superficial lawlessness of pure chance under dominations of law, order, and regularity" (Bell, p. 86).

In his work with probability, Pascal used the arithmetic triangle, which had been known for over six hundred years. His work with the triangle and the new properties he discovered about it, led to it being known as Pascal's triangle (Boyer and Merzbach, p. 404).

All of his life Pascal had medical problems ranging from digestive tract problems to a temporary paralysis (Bell, p. 79) and bad teeth to insomnia. In 1658, he experienced his worst illness ever. He had continuous headaches and was only able to sleep for small amounts of time. After four years of this discomfort, he gave his house to a poor family as an act of self-denial. He then went to live with his married sister. On August 19, 1662, with the onset of convulsions, Blaise Pascal died (Bell, p. 85).

Chapter Twelve Areas and Volumes of Solids

ARCHIMEDES
(287 B.C. - 212 B.C.)

Archimedes is the topic of this section as he contributed much to solid geometry via his formulas for area and volume of many figures.

There was more imagination in the head of Archimedes than in that of Homer. -Voltaire (Boyer and Merzbach, p. 137)

Archimedes was considered the greatest intellect of antiquity (Bell, p. 19). He studied at the University of Alexandria but spent most of his life in his birth place, Syracuse, on the island of Sicily. Since he was a member of the aristocracy, he was able to spend much time concentrating on his studies and thus produced many results involving mathematics and mechanics (Dodge, p. 82). Included in these accomplishments is the origination of statics and hydrodynamics, two branches of physics (Hood, p. 403).

There are fifty-three propositions included in the two book work entitled On the Sphere and Cylinder. Included in this work were results he had obtained such as the surface of a sphere is four times the area of the great circle of the sphere (Burton, p. 218). He applied the general methods he used for finding areas of curvilinear plane figures and volumes bounded by curved surfaces to special instances. These special instances included the circle, the sphere, any segment of a parabola, cylinders, cones, parabolids, hyperboloids, and spheroids (Bell, p. 30). In 1906 his work entitled, The Method, was rediscovered (Boyer and Merzbach, p. 153). This work describes Archimedes' approach to integration, a method that would "enable one to begin to investigate some problems in mathematics by means of mechanics" (Boyer, p. 380).

Other contributions made by Archimedes include his approximation for pi, three and one seventh, using inscribed and circumscribed regular polygons. He continually doubled the number of sides of these polygons until he obtained the perimeters of polygons with ninety six sides (von Baravalle, p. 150). He also used his abilities to build several interesting items including the Archimedian screw which is still used today. It is said that he invented this during a trip to Egypt in order to raise canal water over levees into irrigated fields (Burton, p. 215).

Many interesting stories are told about the methods Archimeded used to approach problems. He was known to use any means whatsoever to discover the result of what he was thinking about. He did not confine himself to tangible things, nor did he restrict himself to the "rules" set down by others, such as Plato's (429 - 348 B.C.) straightedge and compass limitations (Dodge, p. 82). Disregarding Plato's restrictions on constructions, Archimedes said he could trisect an angle. All he needed was a compass and a straightedge with two marks on it (Posamentier and Stepelman, p. 158).

It is said that Archimedes would work anywhere with whatever tools were available. He would do work in the ashes of a fire with a stick or after anointing himself with oil following a bath he used his fingernail on his oily skin (Dodge, p. 82). Other stories about Archimedes follow.

The story is told that King Hieron had a crown made and asked Archimedes to prove that it was pure gold without melting it. While taking a bath, Archimedes discovered the first law of hydrostatics, that a body emerged in a liquid displaces its own volume. Upon this discovery, he leapt from the bath and ran down the streets of Syracuse yelling "Eureka, eureka!" (I have found it, I have found it!). According to the story, the crown has been adulterated with silver by the goldsmith that made it (Dodge, p. 83).

Archimedes was also quoted as saying "Give me a place to stand on and I will move the earth." This was in reference to his discovery of the laws of levers (Bell, pp. 29 - 30).

Archimedes also invented war machines used during the Second Punic War. He used catapults to hurl stones; ropes, pulleys, and hooks to raise and smash the Roman ships; and even some type of device that set fire to the ships (Boyer and Merzbach, p. 137). Because they believed that they had won, the Syracusans were celebrating and did not realize they were being attacked from behind. During this attack is when Archimedes is supposed to have died, despite the order that his life be spared. One story accounts that a soldier stepped on a diagram Archimedes was working on in the dust. Upon being told "Don't disturb my circles!" the soldier killed him. Another version of the story is that Archimedes told a soldier to wait until he had finished a problem and he would go with him. But this soldier did not like being told what to do and killed the seventy five year old man (Bell, p. 84).

Another story relates the idea that Archimedes was killed by looters who mistook his polished brass astronomical instruments for gold and killed him for them. No matter how he died, a monument was erected in his honor. Since he had told friends that he wanted his tomb to bear the figure of a sphere inscribed in a right cylinder, they did this. This figure was in memory of his discovery of the relation of between these two bodies, that the volume of the sphere is equal to two-thirds that of the circumscribing cylinder (Burton, p. 217).

Chapter Thirteen Coordinate Geometry

RENE DESCARTES
(1596 - 1650)

Descartes is known for creating analytic geometry which relates algebra and geometry (Adele, p. 461). It is said that analytical geometry is the "royal road" to geometry that Euclid told Ptolemy did not exist (Eves, p. 180).

> [Analytic geometry], far more than any of his metaphysical speculations, immortalized the name of Descartes, and constitutes the greatest single step ever made in the progress of the exact sciences. -John Stuart Mill (Bell, p. 35)

Descartes was born into a welloff family, and thus received a thorough education. But since he was always sickly he did not start school until he was eight. At this time he was granted special privileges, his attendance was not required at lectures and he was allowed to lie in bed as long as he desired each morning (Burton, p. 342). He studied law without any real interest in it. Tired of school and books, he traveled with various military campaigns where he met many leading scholars of his time. He eventually gained the title "father of modern philosophy," changed the scientific world view, and established a new branch of mathematics (Boyer and Merzbach, p. 375).

He became seriously interested in mathematics after spending the cold winter of 1619 with the Bavarian army. During this time he would lie in bed until ten o'clock in the morning working on problems (Boyer and Merzbach, p. 376).

Stories are told of how Descartes came to think of analytic geometry and another involving Descartes and Queen Christina of Sweden gives us a famous pun.

One story says that the idea for analytic geometry came to Descartes one day when he was watching a fly crawling on the ceiling in the corner of his room. The question of how to express the path of this fly in terms of distance from the adjacent wall posed itself to him. Thus the start for analytic geometry, or so the story goes (Burton, p. 347). On St. Martin's Eve, November 10, 1619, Descartes had a series of three dreams which some say contributed to the discovery of analytic geometry. In the first dream he was blown by evil winds from the security of his church or college toward a third source which the wind could not budge. In the second dream he was observing a terrific storm with the unsuperstitious eye of science. He noted the storm, saw it for what it was and realized it could do him no harm. In the third dream he was reciting the poem of Ausonius, beginning "What way of life shall I follow?" From these dreams, Descartes claims that the magic key to unlock the treasure house of nature had been revealed to him in the second dream. Some claim this key was the combination of algebra and geometry (Bell, p. 39 - 40).

It is said that the Queen of Sweden had an extreme desire for knowledge. To feed this desire, she requested that Descartes leave his warm home in Holland to teach her mathematics. This was late in Descartes' life, he was ill and normally stayed in bed until noon each day. The Queen demanded that Descartes give her lessons in a drafty library of her castle at five in the morning, before her daily horseback ride. (Some accounts claim that Descartes lived in the castle which was damp, drafty, and cold others say that he lived across town and had to make the trip to her castle in one of the worst winters Sweden had seen.) The early hours and the Stockholm weather were too much for Descartes, he died within four months from inflammation of the lungs (pneumonia). Thus comes to us the pun "One should never put Descartes before the horse" (Dodge, p. 103).

He was buried in a cemetery where infants that died before baptism occurred were lain to rest. Fifteen years later his body was returned to France. A wonderful monument was built in the Church of Saint Genevieve in his memory. Since his writings were banned by the Church and universities, the court banned any type of celebration or honor for him. It is said that the man who arranged the return of the remains kept Descartes' right hand as a memento (Burton, p. 345).

Descartes published his work on analytic geometry in a book entitled La Geometrie. Descartes left several things incomplete in this book "in order to give others the pleasure of discovering [it] for themselves." One would find statements such as "I did not undertake to say everything" or "It already wearies me to write so much about it" in his book (Burton, p. 354). This book was divided into three parts, the title of the first is "Problems Which Can Be Constructed by Means of Circles and Straight Lines Only" (Burton, p. 348). The second part of his work was "On the Nature of Curved Lines" and the last section deals mainly with algebra, discussing the nature of equations and the principles underlying their solution. Included in this last section was a rule known as "Descartes' rule of signs" this formula was used by Descartes to determine the possible number of positive roots of a polynomial (Burton, pp. 350 and 352).

Other credits given to Descartes include his ability to trisect an angle using a parabola and a circle (Habegger, p. 201) and the origination of the words "real" and "imaginary" (Baumgart, p. 346). Descartes is known for improving the symbolism used in algebra such as the use of exponents (Sloyen, p. 310) and the use of letters at the beginning of the alphabet for quantities and those near the end for the unknown (Burton, p. 336).

One should note that some sources also give Pierre de Fermat (1601 - 1665) credit for contributing to the development of analytic geometry but Descartes is generally the one who receives the credit (Eves, p. 180).

Chapter Fourteen Transformations

FLEIX KLEIN
(1849 - 1925)

Work with groups led Klein to the Erlanger program. This was a method of classifying geometries based on properties that are unchanged under a particular group of transformations (Boyer and Merzbach, p. 613).

When asked by a perplexed student the secret of mathematical discovery, Klein replied: "You must have a problem. Choose one definite objective and drive ahead toward it. You may never reach your goal, but you will find something of interest on the way." (Bell, p. 419)

In 1871 and 1873 Klein published two monographs entitled On the So-Called Non-Euclidean Geometry. In these works he provided models for Lobachevskian geometry and two types of Riemannian geometry (Burton, p. 568).

At twenty three years of age, Klein was made full professor at the University of Erlangen, after refusing an offer from Johns Hopkins to become a professor of geometry at Oxford University (Burton, p. 588). As was the custom of the time in 1872, he gave an inaugural lecture to the philosophical faculty and the senate. This address came to be known as the Erlanger Program for geometry (Burton, pp. 568 - 569). This program characterizes geometries by using properties that remain invariant, or fixed, under a particular group of transformations (Adele, p. 462). Not only does this program give one the means to organize the known geometries, it also indicates how new geometries could be defined (Burton, p. 569). One would pick a fundamental element of the geometry, the totality of these elements, and then the group of transformations that would be performed on this totality of elements (Eves, p. 188).

Examples of geometries and their transformation group follow. For plane Euclidean metric geometry, the group of transformations is the set of all rotations and translations in the plane. In plane projective geometry, the group of transformations is the set of all planar projective transformations. For topology, the group of transformations is the set of all topological transformations (Eves, p. 188).

It should be noted that is was Arthur Cayley (1821 - 1895) who set the ground work for Klein to discover that Euclid's, Lobachevsky's, and Riemann's geometries were all just special cases of a more general kind of geometry (Bell, p. 379).

In 1886, Klein was asked to go to the University of Gottingen to be their leading research mathematician. His work is said to have saved the university from a mathematical low. Supposedly he even raised the university to a level greater than it had been in the days of Gauss. It is said that his "favorite pupil" was an Englishwoman Grace Chisholm Young. In England, women were not admitted to graduate schools. Young became the first woman to be granted a German doctoral degree (1895), in any subject whatever, through the regular examination process (Burton, p. 568).

Other contributions made by Klein include the names "elliptic geometry" and "hyperbolic geometry" for the hypothesis of the obtuse and acute angle used by Saccheri (1697 - 1733) and Lambert (1728 - 1777) during their work with the parallel postulate. He worked with topology and is today recognized by the one-sided surface known as the Klein bottle. After his death, his classic history of mathematics was published. It showed how familiar he was with all aspects of mathematics (Boyer and Merzbach, pp. 613 - 614).

Klein was noted as being a very influential teacher. He not only gave inspiring lectures, he was also concerned with the teaching of mathematics at many levels. He exerted a strong influence in many pedagogical circles (Boyer and Merzbach, p. 614). In 1893, he attended the International Scientific Congress at the World's Fair in Chicago. After it, he gave a series of lectures at Northwestern University. This series was the first colloquium of American mathematicians (Burton, p. 588).

In 1906, Maurice Frechet (1878 - 1973) began the study of abstract spaces. From his studies arose a few general items that did not fit into the Kleinian scheme. Because the mathematicians still wanted to call this area a geometry, the definition of space changed. Space became a set of objects together with a set of relations in which the objects are involved. The geometry became the theory of such a space. This work caused the lines between geometry and other areas of mathematics to become blurred, but the Klein program has withstood the test of time (Eves, p. 190).

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